Outline   Combinatorial Approach to Simplicial Sets

- Motivation
- Simplicial complex → ~~delta set~~ → abstract simplicial set → delta set → S-Set
  asd
- Simplicial data structures

~~Homotopy~~

Motivation

$$\text{Top} \rightleftarrows \text{S.Set} \rightleftarrows \text{Categories}$$

geometric n-simplex

$$[v_0, v_1, v_2, \dots, v_n] := \left\{ p : p = \sum_{i=0}^{n} t_i v_i \,,\ t_i \geq 0 \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}$$

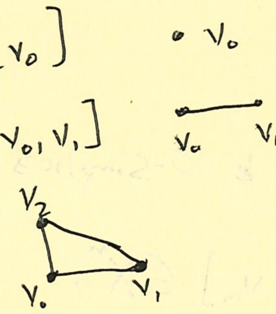
with $v_0, v_1, \dots, v_n$ geometrically independent (i.e.

$v_1 - v_0, v_2 - v_0, v_3 - v_0, \dots, v_n - v_0$ are linearly independent.)

0-simplex $[v_0]$

1-simplex $[v_0, v_1]$

2-Simplex



} faces

Vertices determine the n-simplex.

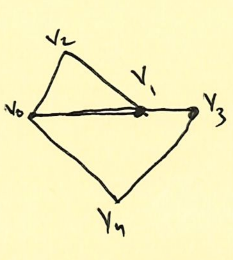
$v_i = e_i$ gives geometric n-simplex where e_i are basis for $\mathbb{R}^{n+1}$

build simplicial complex in $\mathbb{R}^N$

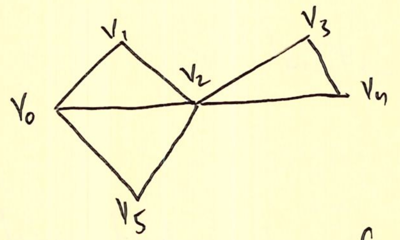
X is simplicial complex if

1. X contains simplices
2. Every face of a simplex in X is also in X
3. For any two simplices $\sigma_1, \sigma_2 \in X$, either $\sigma_1 \cap \sigma_2 = \emptyset$
   or $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2

So,



is not a complex

but  is

How to organise this information?

X^k = collection of k-simplices

$X^0 = \cancel{\{v_1\}\{v_2\}\{v_3\}\{v_4\},\{v_5\}}$ $[v_i]_{i \in I}$

$X^1 = \{ [v_0, v_1], [v_1, v_2], [v_0, v_2], [v_0, v_5], [v_2, v_5], [v_2, v_3], [v_2, v_4], [v_3, v_4]$

$X^2 = [v_0, v_1, v_2], [v_0, v_2, v_5], [v_2, v_3, v_4]$

Abstract Simplicial Complex

(X, V) with $X = X^0 \cup X^1 \cup X^2 \cup \dots$ is a collection of subsets of V

where $X^0 = \{ \{v\} : v \in V \}$ where $X^k \subset X$,

$\forall\, x \in X^k$, $|x| = k+1$ and any $(j+1)$-element subset of an element of X^k is an element of X^j

Simplicial map

$f : (X, V) \to (Y, W)$ maps k 0-simplices to 0-simplices s.t. for $[v_{i_0}, v_{i_1}, \dots, v_{i_n}] \in X^n$,

$[f(v_{i_0}), f(v_{i_1}), f(v_{i_2}), \dots, f(v_{i_n})] \in Y$

e.g. inclusion

e.g. $v_0 \mapsto v_0$, $v_1 \mapsto v_2$, $v_2 \mapsto v_2$ + all else identity

$X^0 \longmapsto X^0$

$[v_0, v_2] \longmapsto [v_0, v_2]$, $[v_0, v_1] \mapsto [v_0, v_2]$

$[v_1, v_2] \longmapsto [v_2]$

Now assume V is ordered. So now,

$[v_0, v_1, \dots, v_n]$ is n-simplex if $i<j$, then $v_i < v_j$

$|\Delta^n| = [0,1,\dots,n]$ and each k-face of $|\Delta^n|$ is made from $k+1$-subset of $\{0,1,2,\dots,n\}$, ie.

$[i_0, i_1, \dots, i_k]$ with $0 \le i_0 < i_1 < i_2 < \dots < i_{k-1} < i_k \le n$

e.g. faces of $[0,1,2]$ are $[0,1]$, $[0,2]$, $[1,2]$, $[1]$, $[0]$, $[2]$

n-simplex will have $n+1$ faces $+ (n+1)n + n(n-1) + \dots + 1\cdot 2\cdot 3\cdot 4 \cdots n$

encode this in face maps $d_0, d_1, \dots, d_0$

$$d_j^{(k)} : X^k \longrightarrow X^{k-1} \quad \text{with} \quad 0 \le j \le k$$

sending $[i_0, i_1, \dots, i_k] \longmapsto [i_0, i_1, \dots, \hat{i_j}, \dots, i_k]$

$$[0,1,2,3,4,5,6] \xrightarrow{d_6^{(6)}} [0,1,2,3,4,5] \xrightarrow{d_5^{(5)}} [0,1,2,3,4]$$
$$\xrightarrow{d_2^{(4)}} [0,1,3,4] \xrightarrow{d_0^{(3)}} [1,3,4]$$

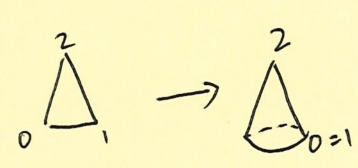
<u>Prop</u> Between any two simplices of different dimensions, $\exists!$ sequence of chain maps

Note: if $i<j$ $\quad d_i d_j = d_{j-1} d_i$

A <u>delta set</u> X consists of a sequence of sets $X_0, X_1, X_2, \dots$ and for each $n \geq 0$, maps $d_i : X_{n+1} \to X_n$ for each i with $0 \le i \le n+1$ such that $d_i d_j = d_{j-1} d_i$ whenever $i<j$

e.g abstract simplicial complex.

e.g.



$X_0 = \{[0]=[1], [2]\}$

$X_1 = \{[0,2]=[1,2], [0,1]\}$

$X_2 = \{[0,1,2]\} = \{|\Delta^2|\}$

$d_2[0,1,2] = [0,1]$, $d_0[0,1] = d_1[0,1] = [0] = [1]$

Note: Delta sets are not determined by their vertices

Sneak peak

$f : |\Delta^2| \to |\Delta^1|$

$X_0 \mapsto Y_0$
$X_1 \mapsto Y_1$
$X_2 \mapsto Y_2$ etc.

with $f(0) = 0$, $f(1) = f(2) = 2$

Sends $X_2 \mapsto \emptyset$

Correction: send $[0,1,2] \mapsto [0,2,2]$

$\hat{\Delta}$ = Category

$Obj(\hat{\Delta}) = \underline{n} = \{0,1,2,\dots,n\}$

$f \in Hom_{\hat{\Delta}}(\underline{n}, \underline{m})$ is ~~weakly~~ strictly order preserving

$i < j \Rightarrow f(i) < f(j)$

Observ: for $m < n$, then $Hom_{\hat{\Delta}}(\underline{n}, \underline{m}) = \emptyset$. Also, no constant map

$\{0,1,2,\dots,n-1\} \xhookrightarrow{f_i} \{0,1,2,\dots,n\}$ $n+1$ such maps

$f_i(r) = \begin{cases} r & r < i \\ r+1 & r \geq i \end{cases}$

$0 \mapsto 0$
$1 \mapsto 1$
$2 \mapsto 2$
$\vdots$
$i \mapsto i+1$
$i+1 \mapsto i+2$
$\vdots$
$n-1 \mapsto n$

$\{0,1,2,\dots,n-1\} \xrightarrow{\phi_i} \{0,1,2,\dots,\hat{i},\dots,n\}$

~~$\phi_i : \underline{n} \to \underline{n+1}$~~

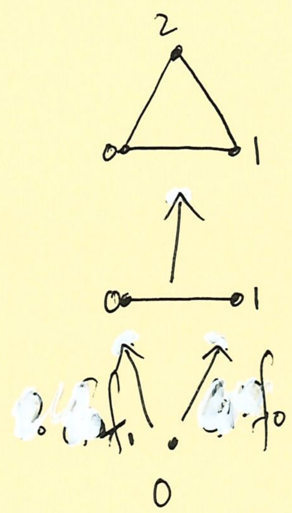
exercise: let $\hat{\Delta}^{op}$ be opposite category with ~~$f_i \mapsto \delta_i$~~ $f_i \mapsto \delta_i$

then $d_i d_j = d_{j-1} d_i$ (b/c $f_j f_i = f_{j-1} f_i$)

Def^n Delta set is a functor $X: \hat{\Delta}^{op} \to \text{Set}$

$\underline{n} \mapsto X_n$

for $m < n$, $\underline{m} \to \underline{n} \mapsto X_n \xrightarrow{d} X_m$



$\underline{0} \mapsto \{[v_0],[v_1],[v_2],[v_3],[v_4],[v_5]\}$

$\underline{1} \mapsto \{[v_0,v_1],[v_0,v_2],[v_0,v_5],[v_2,v_5],$
$[v_2,v_3],[v_3,v_4],[v_2,v_4],[v_1,v_2],\}$

$\underline{2} \mapsto [v_0,v_1,v_2],[v_0,v_2,v_5],[v_2,v_3,v_4]$

A map $X_\bullet \xrightarrow{\eta} Y_\bullet$ is a natural transformation

~~for, $d_i: X_n \to X_{n-1}$ X_n~~

for $\delta_i : \underline{n} \to \underline{n-1}$,

$$\begin{tikzcd} X(\underline{n}) \arrow[r, "\eta_n"] \arrow[d, "X(\delta_i)"'] & Y(\underline{n}) \arrow[d, "Y(\delta_i)"] \\ X(\underline{n-1}) \arrow[r, "\eta_{n-1}"] & Y(\underline{n-1}) \end{tikzcd}$$

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "d_i"'] & Y_n \arrow[d, "d_i"] \\ X_{n-1} \arrow[r] & Y_{n-1} \end{tikzcd}$$

let Δ be category with $\text{Obj}(\Delta) = \underline{n}$

and $\text{Hom}_\Delta(\underline{m},\underline{n})$ = order preserving maps

$i \le j$, $f(i) \le f(j)$

$n+1 = |\text{Hom}_\Delta(\underline{0},\underline{n})|$, $1 = |\text{Hom}_\Delta(\underline{n},\underline{0})|$,

$\frac{(n+2)(n+1)}{2} = |\text{Hom}_\Delta(\underline{1},\underline{n})|$, $n+2 = |\text{Hom}_\Delta(\underline{n},\underline{1})|$

note: now constant maps.

for $m \leq n$, $\mathrm{Hom}_{\Delta}(\underline{n}, \underline{m}) \neq \emptyset$ $n-1$ such maps

$\{1, 2, \dots, n\} \xrightarrow{g_i} \{0, 1, 2, \dots, n-1\}$ unique surjective.

$$g_i(r) = \begin{cases} r & r \leq i \\ r-1 & r > i \end{cases}$$

$$g_i^{-1}(i) = \{i, i+1\}$$

$0 \mapsto 0$
$1 \mapsto 1$
$2 \mapsto 2$
$\vdots$
$i \mapsto i$
$i+1 \mapsto i$
$i+2 \mapsto i+1$
$\vdots$
$n-1 \mapsto n-2$
$n \mapsto n-1$

Prop Any order preserving map $\underline{n} \to \underline{m}$ is a composition of g_i and f_j

$\cancel{D_i D_j = D_{j-1} D_i}$

$f_j f_i = f_i f_{j-1}$ $i < j$

$g_j f_i = f_i g_{j-1}$ $i < j$

$g_j f_j = id_{\underline{n-1}}$, $s_j f_{j+1} = id_{\underline{n-1}}$

$s_j s_i = s_i s_{j+1}$ $i \leq j$

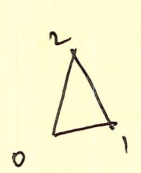
$g_i f_i = g_j f_{i-1}$ $i > j+1$

$$\begin{tikzcd} & \underline{n} \arrow[dr, "f_j^{(n+1)}"] & \\ \underline{n-1} \arrow[ur, "f_i^{(n)}"] \arrow[dr, "f_{j-1}^{(n)}"'] & & \underline{n+1} \\ & \underline{n} \arrow[ur, "f_i^{(n+1)}"'] & \end{tikzcd}$$

A Simplicial set is the functor $X : \Delta^{op} \to Set$

$s_i : X_n \to X_{n+1}$ n such maps

$s_i [0, 1, 2, \dots, n] = [0, 1, 2, \dots, i, i, i+1, \dots, n]$



$X_0 = \{[0], [1], [2]\}$

$X_1 = \{[0,1], [0,2], [1,2], [0,0], [1,1], [2,2]\}$

$X_2 = \{[0,1,2], [0,0,1], [0,0,2], [0,1,1], [0,2,2], [1,2,2], [0,0,0], [1,1,1], [2,2,2], [1,1,2]\}$

let
$$\begin{tikzcd} \Delta^{op} \arrow[r, bend left=50, "X"] \arrow[r, bend right=50, "Y"'] \arrow[r, phantom, "\Downarrow F"] & Set \end{tikzcd}$$
then
$$\begin{tikzcd} X_n \arrow[r, "F_n"] \arrow[d, "s_i"'] & Y_n \arrow[d, "s_i"] \\ X_{n+1} \arrow[r, "F_{n+1}"'] & Y_{n+1} \end{tikzcd}$$
$\rightsquigarrow \int_* G \longrightarrow H$

$\cancel{F_*(G) = F_*(X \times X) = F_*\left(\bigcup_k X_k \times \bigcup_i X_i\right)}$

$= \cancel{\bigcup_k F(X_k) \times \bigcup_i F(X_i)}$

$G = \mathcal{G}(X) = \bigcup_{i,j} X_i \times X_j \quad |i-j| \leq 1$

$= (X_0 \times X_0) \cup (X_0 \times X_1) \cup X_1 \times X_0$
$\cup (X_1 \times X_2) \cancel{\times} X_2 \times X_1 \cup (X_1 \times X_1)$
$\cup \dots$

$\cancel{\hat{F}_* : P(G) \longrightarrow P(H) =}$

$\mathcal{G}(F_n)\{G\} = \bigcup_{i,j} F_i X_i \times F_j X_j$

$\cancel{\hat{F}_* S_G(A) \in \hat{F}_* X_i \quad \text{for some } i}$

$\cancel{\in Y_i}$

$\underline{S_H} \hat{F}(A) = \quad \cancel{F_* S_G(A) = F_* S_G\{(y_i, y_j)\}}$

Category of Generalized Graphs.

A (generalized) graph is a set G with functions

$$s: P(G) \to G$$
$$t: P(G) \to G$$

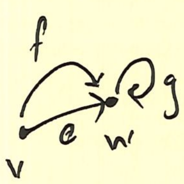
A morphism $f \in \mathrm{Hom}_{\mathrm{Graph}}(G, H)$ b/w graphs G and H

is a map $f: G \to H$ s.t. $f s_G = s_H \hat{f}$ and $f t_G = t_H \hat{f}$

$$\begin{tikzcd} P(G) \arrow[r, "\hat{f}"] \arrow[d, "s_G, t_G"'] & P(H) \arrow[d, "s_H, t_H"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

a vertex of G is an element $v \in G$ s.t.

$$s(\{v\}) = t(\{v\}) = v$$

ex 

$G = \{v, e, w, f, g\}$

$P(G) = \{\emptyset, \{v\}, \{e\}, \{w\}, \{f\}, \{g\}, \{v,e\}, \{v,w\},$
$\{v,f\}, \{v,g\}, \{e,w\}, \{e,f\}, \{e,g\}, \{w,f\}, \{w,g\},$
$\{f,g\}, \{v,e,w\}, \{v,e,f\}, \{v,e,g\}, \{e,w,f\}$
$\{e,w,g\}, \{w,f,g\}, \{v,e,w,f\}, \{v,e,w,g\},$
$\{e,w,f,g\}, \{v,e,w,f,g\}, \{e,f,g\}, \cancel{\{w,e,f\}}$
$\{v,f,g\}, \{v,w,f\}, \{v,w,g\}\}$